Journal of Statistical Physics, Vol. 8, No. 2, 1973

A Particle Picture of "Tunneling" and the Nature of "Photon"-Matter Interaction[1]

V. J. Lee[2,3]

Received January 5, 1973

Brownian motion is employed as a model to formulate a probabilistic statement of the Heisenberg uncertainty relationship. By means of the latter, a tunneling formula identical to that obtained from the WBK method is derived from the particle picture. The nature of photon-matter interaction is discussed in the light of the quantal Brownian motion.

KEY WORDS: Tunneling; Poisson process; uncertainty principle; photon.

1. INTRODUCTION

A derivation of the Schrödinger equation from Newtonian mechanics has been eminently achieved by means of a statistical model.[1] The electron is regarded as a *point particle* of mass m constantly undergoing a Brownian motion with the diffusion coefficient of $(\hbar/2m)$, where $\hbar$ is the Planck's constant h divided by 2π. As a consequence of the stochastic model, we can interpret the uncertainty action s, defined to be the product of the root mean square

Research supported under NSF Grant #GP-12408.

[1] Contribution No. 3080 from the Department of Chemistry, University of California, Los Angeles, California.

[2] Department of Chemistry, University of California, Los Angeles, California.
[3] Currently on sabbatical leave from the University of Missouri-Columbia.

deviation of any canonical conjugate pair $\Delta q_i \cdot \Delta p_i$, as being a result of the quantal Brownian motion. Thus, from Einstein's equation of diffusion,

$$\langle (x - x_0)^2 \rangle = 2Dt \tag{1}$$

it can be easily shown that[1]

$$\langle \Delta x \rangle \langle \Delta p \rangle = \hbar = \bar{s} \tag{2}$$

upon replacing D by $\hbar/2m$. Since Eq. (2) is an equality, we may conjecture that $\hbar$ is the statistical average of the uncertainty action $\bar{s}$.

A small particle performs Brownian motion in a fluid which is caused by collisions with molecules of the fluid. The probability of no collision in the time interval $(0, t)$ is given by

$$P_0(t) = e^{-t/\tau}$$

The random collision process is referred to as the Poisson process. Hence, in this paper we propose the hypothesis that the uncertainty action (of a canonical conjugate pair) of a particle in a quantal Brownian motion is again a Poisson process. Thus we have the following postulate:

> Whatever the value of the uncertainty action s of a particle is, for a small interval of length Δs in the sample space the probability of finding the value of s in Δs is $\lambda\, \Delta s + O(\Delta s)$, where $\lambda > 0$ and $O(\Delta s)$ denotes a zero quantity which is of smaller order of magnitude than Δs, and $1 - \lambda\, \Delta s$ is the probability of not finding the value of s in Δs.[2]

With this postulate we shall formulate the probability density function $\rho(s)$ in the sample space $s = \{s : \hbar/2 \leqslant s < \infty\}$. It goes without saying that the lower bound $s \geqslant \hbar/2$ is specified in accordance with the Heisenberg uncertainty principle.

Since $\rho(s)$ is the probability density function, $\rho(s)\, \Delta s$ is the probability of finding the value of the uncertainty action of the particle in the interval $(s, s + \Delta s)$. Let us consider two contiguous intervals, $(\hbar/2, s)$ and $(s, s + \Delta s)$ in the sample space. The value of the uncertainty action of the particle must *not* be in $(\hbar/2, s)$ and be in $(s, s + \Delta s)$. Let $P_0(s)$ be the probability that the value of the uncertainty action of the particle is *not* in the interval $(\hbar/2, s)$. From this conjecture and the aforementioned postulate, we obtain

$$\rho(s)\, \Delta s = P_0(s) \lambda\, \Delta s \tag{3}$$

and

$$P_0(s + \Delta s) = P_0(s)(1 - \lambda\, \Delta s) \tag{4}$$

As $\Delta s \to 0$, Eq. (4) reduces to

$$P_0'(s) = -\lambda P_0(s) \tag{5}$$

It should be pointed out that Eq. (5) is of the same form as that obtained from the sample space $s' = s - \hbar/2$ in the standard text on probability.[2] With this consideration and the usual condition,

$$P_0(s')|_{s'=0} = 1 \tag{6}$$

we obtain

$$P_0(s) = e^{-\lambda(s-\hbar/2)} \tag{7}$$

Substituting Eq. (7) into Eq. (3), we have

$$\rho(s) = \begin{cases} \lambda e^{-\lambda(s-\hbar/2)} & \text{for} \quad s \geqslant \hbar/2 \\ 0 & \text{elsewhere} \end{cases} \tag{8}$$

We note that $\rho(s)$, as given by Eq. (8), has been normalized; thus,

$$\int_{\hbar/2}^{\infty} -e^{\lambda(s-\hbar/2)}\lambda \, ds = 1 \tag{9}$$

Equation (9) means physically that a particle has the uncertainty action $\hbar/2 \leqslant s \leqslant \infty$.

To determine λ, we apply the condition that the statistical *average* uncertainty action of a particle performing quantal Brownian motion is $\hbar$, as indicated by Eq. (2). Thus,

$$\bar{s} = \hbar = \int_{\hbar/2}^{\infty} s e^{-\lambda(s-\hbar/2)}\lambda \, ds = (\hbar/2) + (1/\lambda) \tag{10}$$

Hence we obtain

$$\lambda = 2/\hbar \tag{11}$$

Substituting Eq. (11) into Eqs. (7) and (8), respectively, we obtain the following interesting equations:

$$P_0(s) = \begin{cases} e^{-2(s-\hbar/2)/\hbar} & \text{for} \quad s \geqslant \hbar/2 \\ 1 & \text{elsewhere} \end{cases} \tag{12}$$

$$\rho(s) = \begin{cases} (2/\hbar)\, e^{-2(s-\hbar/2)/\hbar} & \text{for} \quad s \geqslant \hbar/2 \\ 0 & \text{elsewhere} \end{cases} \tag{13}$$

Equation (12) says that the probability of finding a particle with the uncertainty action equal to $\hbar$ is 0.632. Equation (13) may be regarded as the probabilistic statement of Heisenberg's uncertainty relationship.

2. APPLICATION TO "TUNNELING"

Every potential "hill" can be considered to have a given amount of action. To illustrate this point, let us consider

$$V(x) > E \qquad \text{for} \quad x_1 \leqslant x \leqslant x_2 \tag{14}$$

The *action* of the potential "hill" with respect to the particle with energy E, $s(V, E)$, is given by

$$s(V, E) = \int_{x_1}^{x_2} p\,dx = (2m)^{1/2} \int_{x_1}^{x_2} (V - E)^{1/2}\,dx \tag{15}$$

The particle must have $s(V, E)$ additional action before it can reach x_2 from x_1. To find the particle at x_2, it must have a *positive* uncertainty action of the amount $s(V, E) + \hbar/2$. Hence, the probability of finding the particle at x_2 is, at most,

$$P_0(s + \hbar/2) = \exp[-2(2m/\hbar^2)^{1/2} \int_{x_1}^{x_2} (V - E)^{1/2}\,dx] \tag{16}$$

The qualifying "at most" is because of the fact that the quantal Brownian motion may increase as well as decrease the action of a particle. But by the definition of s as a product of root mean square deviation. s is always positive. *A priori* the preexponential factor of Eq. (16) should be 1/4 (because the increase and decrease of the action of the particle are at equal chance, as well as moving in the two directions of x). Finally, it can be concluded that from this viewpoint, the particle *goes over* the potential barrier by its quantal Brownian motion rather than penetrates the potential hill via tunneling. This point of view of tunneling as a climb over the potential barrier is similar to the one given by Cohen.[3]

3. DISCUSSION OF PHOTON–MATTER INTERACTION

In accordance with the quantal Brownian motion, the statistical average uncertainty action of an electron is $\hbar$. One can imagine the electron in the hydrogen atom "dancing" in and out or about the proton with an average radius a and an average momentum p such that

$$a \cdot p = \hbar \tag{17}$$

Hence, the kinetic energy of the electron is $\hbar^2/2ma$. The total energy E of the electron is given by

$$E = (\hbar^2/2ma^2) - (e^2/a) \tag{18}$$

Taking a given value such that $dE/da = 0$, one obtains $E = -13.6$ eV.[4] This is the amount of energy required to move the electron from the hydrogen atom. Hence $\bar{s} = \hbar$ is in agreement with a well-known observation in atomic physics.

The Brownian motion of small particles in a fluid is caused by random collisions with molecules of the fluid. But what is the cause of the quantal Brownian motion of an electron?

From the fact that the diffusion constant in quantal Brownian motion is $\hbar/2m$ it is reasonable to speculate that the quantal Brownian motion is caused by "photons" or "virtual photons." Each photon may be considered as carrying an action h ($\epsilon \cdot \tau = h\nu/\nu$) when it interacts with an electron in three-dimensional space. When the electron absorbs a photon, it increases in action by h. But since the electron has 4π directions in three-dimensional space, the distribution of the increase in action in any one direction is $h/4\pi$. Hence, whenever we look at an electron by interacting it with a photon, an uncertainty action of at least $\hbar/2$ is the input to the electron in a given direction. Likewise, an electron can decrease its "content of action" by emitting a photon of energy ϵ in a time interval τ such that $\epsilon \cdot \tau = h$. The decrease of "action" of each of the 4π directions is $h/4\pi$. Consequently, the uncertainty action defined as the product of root mean square derivations of a canonical conjugate pair is again *increased* by $\hbar/2$. Moreover, if this electron–photon interaction is the same as the random collision process in the Brownian motion, the accumulation of the uncertainty action s is naturally a Poisson process.

The hypothesis that photons and matter interact by transfer of an integral action h implies that the interaction time is the wavelength divided by the velocity of light. In photoelectron emission this means that the time lag between the beginning of illumination and the start of the photoelectric current is of the order of 10^{-15} sec if visible light is used. Previous measurements indicate that if such a lag exists it is less than 3×10^{-9} sec.[5] Hence, this hypothesis does not contradict previous experiments.

The above hypothesis on photon–matter interaction can also provide an interpretation for the quantum condition of Bohr and Sommerfeld. Thus an electron in orbital motion can change its angular momentum only such that

$$\oint p\,dq = nh \tag{19}$$

upon interaction with n photons.

The diffraction of electrons (or atoms) may be conjectured as being due to the interaction of electrons of the incoming beam and those of the lattice ions.[6] The condition of diffraction is that the incident electrons must not suffer any energy loss. Hence there is no energy gain by the bonding electrons of the lattice ion. Therefore an exact amount of linear momentum, but in opposite direction, must be exchanged between the incident and scattering electrons. From the analysis in Ref. 6, the net change of linear momentum normal to the surface in the positive z direction is $2 \mid p \mid \sin \theta$. Consequently, the scattering electron changes its momentum Δp_z by $-2 \mid p \mid \sin \theta$. But the movement of the bonding electrons of the lattice ion in the z direction is confined between the two neighboring ions with a distance d in the z direction (otherwise it is no longer regarded as a bonding electron). If the bonding electron is capable of interacting with photons in the z direction, its linear momentum $\mathbf{p}_z$ must be able to change by $\Delta \mathbf{p}_z$, whose scalar product with $\mathbf{d}$ must satisfy the relation

$$\Delta \mathbf{p}_z \cdot \mathbf{d} = nh \tag{20}$$

since the only integral action h is involved in such an interaction. Now, substituting $\Delta \mathbf{p}_z = -2 \mid p \mid \sin \theta$ and $\mathbf{d} = -d$ into Eq. (20), we obtain

$$2d \mid p \mid \sin \theta = nh \tag{21}$$

Hence the incident electrons are elastically scattered in specific directions according to their momentum $\mathbf{p}$. Thus the phenomenon of electron diffraction can be explained in the particle picture. But the particle–particle scattering interaction involves an action transfer medium—the photons.

Because electron transfer by tunneling occurs instantaneously and gives rise to the electronic Frank–Condon principle,[7] it is logical to assume that particle tunneling arises from interaction with photons or virtual photons. Naturally, if the interaction is a random process, it is not accidental that Eq. (12) leads to the tunneling probability relationship, Eq. (16).

In view of the above discussion, it is plausible that the "hidden variable" of quantal Brownian motion is photons or virtual photons of the electromagnetic field. The photon–matter and matter–matter interactions *via photons* give rise to the phenomena of uncertainty action, tunneling, the quantum conditions, particle diffraction, and the quantal Brownian motion. The amplitude (i.e., the square root) of the probability density of the latter obeys the Schrödinger equation as shown in Ref. 1.

ACKNOWLEDGMENT

The author is very grateful to Professors Howard Reiss and Willard F. Libby for discussions pertaining to this note; he is also grateful for the financial support rendered to him during his sabbatical stay at the University of California, Los Angeles.

REFERENCES

1. E. Nelson, *Phys. Rev.* **150**:1079 (1966), and references therein; J. Ogunlana, *J. Stat. Phys.* 4(2/3):217 (1972).
2. W. Feller, *An Introduction to Probability Theory and Its Application*, Vol. 1, 3rd ed., Wiley, New York (1968), pp. 446–7.
3. B. L. Cohen, *Am. J. Phys.* **33**:97 (1965).
4. A. Messiah, *Quantum Mechanics*, Wiley, New York (1961), Vol. 1, p. 414.
5. E. O. Lawrence and J. W. Beams, *Phys. Rev.* **29**:903 (1927).
6. A. Lande, *From Dualism to Unity in Quantum Physics*, Cambridge University Press (1960), pp. 61–63; W. Yourgrau and A. van der Mere, *Introduction to "Perspectives in Quantum Theory,"* MIT Press (1971).
7. W. F. Libby, in *Topics in Modern Physics—A Tribute to E. U. Condon*, ed. by W. E. Brittin and H. Odabasi, Colorado Associated University Press, Boulder (1971), pp. 205–218.